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Stationary, transcritical channel flow

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A compact (streamwise scale small compared with characteristic length) pressure distribution, which models a ship and is equivalent to a compact bottom deformation of cross-sectional area A, exerts a net vertical force $\rho g A$ on, and advances with speed U over, the free surface of a shallow canal of upstream depth H. The hypotheses of weak dispersion, weak nonlinearity and steady, two-dimensional flow in the reference frame of the force yield, through a generalization of Rayleigh's (1876) formulation of the (free) solitary-wave problem, a cnoidal wave downstream of the force matched to a null solution on the upstream side if $|A|/H^2 < \frac{2}{9}(1-F^2)^{\frac{3}{2}} \ll 1$ (Cole 1980) or a cusped solitary wave if $|A|/H^2 < \frac{4}{9}(F^2-1)^{\frac{3}{2}} \ll 1$, where $F \equiv U/(gH)^{\frac{1}{2}}$ is the Froude number. The hypothesis of steady flow presumably fails in the transcritical range $1-(9A/2H^2)^{\frac{2}{3}} < F^2 < 1+(9A/4H^2)^{\frac{2}{3}}$, at least under the restrictions of weak dispersion and weak nonlinearity. Comparisons with experiment and numerical solutions of the nonlinear initial-value problem provide some confirmation of the cusped solitary wave but suggest that the cnoidal wave may be unstable in the absence of dissipation.

1. Introduction

I consider here the transcritical flow induced by a two-dimensional, compact pressure distribution that exerts a net vertical force P (per unit breadth) on, and advances with uniform speed U over, the surface of a perfect fluid of uniform density ρ and upstream depth H, as in the investigations of Huang *et al.* (1982), Wu & Wu (1982), Akylas (1984), Ertekin (1984), Ertekin, Webster & Wehausen (1984), and Cole (1985*b*). It appears from these investigations that if the Froude number

$$F = (gH)^{-\frac{1}{2}}\,U \tag{1.1}$$

is in some transcritical range a periodic sequence of solitons propagates upstream of the force† at supercritical speed, and the flow then is intrinsically unsteady. If $F-1$ is not small breaking occurs, and a bore appears upstream of the force. If $F < 1$ a periodic wavetrain appears behind the force, as in the case of a weak bore (Benjamin & Lighthill 1954). The calculations of Wu & Wu (1982) and Ertekin (1984) suggest the existence of a lower critical value of F, F_1, below which this wavetrain is asymptotically (in time) stationary in the reference frame of the force and the upstream solitons are transient, but the experimental evidence suggests that some unsteadiness may persist for any subcritical F (although it should be emphasized that the theoretical formulations may be valid only for much smaller values of $P/\rho g H^2$ than those realized in the experiments). The calculations of Wu & Wu (1982) also suggest the existence of an upper critical value of F, F_u, above which the disturbance

† I now use 'force' to refer, somewhat loosely, to either the pressure distribution or its integrated vertical component P.

resembles a single solitary wave that is stationary in the moving reference frame of the force, but the upstream-soliton regime in the experiments of Ertekin *et al.* is terminated by breaking and the appearance of an upstream bore.

Theoretical considerations imply that disturbances similar to those induced by a moving force occur for flow over a horizontally compact, transverse bump of cross-sectional area $A = P/\rho g$ ($A < 0$ for a depression) on the bottom of a running stream; cf. Lamb (1932, §§177 and 245–249), Ertekin (1984) and Cole (1985*a*).

Against this background, I postulate the existence, in some range(s) of $\mathbb{F}$, of steady, two-dimensional flow induced by a moving pressure distribution on the surface, or a transverse obstacle on the bottom, of a running stream. The corresponding linear problems were solved originally by Kelvin (1886) and are treated in detail by Lamb (1932, §§245–249). The nonlinear problem for $0 < 1-\mathbb{F} \ll 1$ has been solved by J. D. Cole (1980), who matched the (inner) solution of the linear problem to the (outer) solution of the first-order, nonlinear differential equation that governs the downstream wavetrain and obtained an implicit result for $\mathbb{F}_1$. The nonlinear problem for $0 < \mathbb{F}-1 \ll 1$ has been examined by S. L. Cole (1983), using matched asymptotic expansions; however, her end results are in error in consequence of an incorrect matching condition (S. L. Cole, private communication).

My basic assumptions, in addition to those in the opening sentence, are

$$\frac{h-H}{H} = O(\alpha), \quad \beta \equiv \left(\frac{H}{L}\right)^2 = O(\alpha), \quad \epsilon \equiv \frac{A}{H^2} = O(\alpha^{\frac{3}{2}}), \quad \mathbb{F} = 1+O(\alpha), \quad (1.2a, b, c, d)$$

$$h \sim h(X-Ut) \equiv h(x) \quad (t \uparrow \infty) \quad (1.3)$$

and

$$h(x) \sim H \quad (x \uparrow \infty), \quad (1.4)$$

where h is the local depth, L is a characteristic wavelength, α is a measure of nonlinearity, β is a measure of dispersion, and $X(x)$ is directed upstream in the laboratory (moving) reference frame. The double limit $t\uparrow\infty$, $x\uparrow\infty$ is not, in general, commutative, and H may differ from the initial depth, say h_0, in consequence of a transient surge of elevation $H-h_0$ that moves ahead of P with the relative velocity $(gh_0)^{\frac{1}{2}}-U$ if $\mathbb{F} < 1$ (Benjamin 1970); see §4. However, (1.4) rules out an upstream wavetrain, which hypothesis is consistent with the known results of linearized theory [Lamb 1932, §245; Wurtele 1955] and numerical solutions of the initial-value problem based on the Boussinesq equations for $\mathbb{F} > \mathbb{F}_u$ (Wu & Wu 1982), but is contradicted by the aforementioned experiments and by numerical solutions of the Green–Naghdi equations for $\mathbb{F} < 1$ (Wehausen & Ertekin, private communication) and of the inhomogeneous Korteweg–de Vries equation for $(\mathbb{F}^2-1)/\epsilon^{\frac{2}{3}} = 2$ and -3 (Akylas, private communication). This suggests that, in the absence of dissipation, the solutions to be obtained here may be either unstable (although small dissipation might render them stable) or attainable only after rather long times.

The hypotheses (1.3) and (1.4) permit the reduction of the equations of motion to a nonlinear differential equation for h. I follow Rayleigh's (1876) formulation of the solitary-wave problem for this reduction (§2); alternative formulations follow from the Boussinesq (Wu & Wu 1982), Green–Naghdi (Ertekin 1984; Naghdi & Vongsarnpigoon 1985), or (forced) Korteweg–de Vries (Akylas 1984) equations, from the method of matched asymptotic expansions (Cole 1980), and from Hamilton's principle (Appendix A). The analytical solution of this differential equation, subject to the requirement that the wave drag be non-negative, for $\mathbb{F} < 1$ (§4) yields a cnoidal wavetrain downstream, matched to a null solution ($h = H$) upstream, of the force if and only if (Cole 1980)†

$$\epsilon < \tfrac{2}{9}(1-\mathbb{F}^2)^{\frac{3}{2}}. \tag{1.5}$$

The solution for $\mathbb{F} > 1$ (§5) yields a cusped solitary wave (the cusp is convex/concave for $A \gtrless 0$) if and only if

$$\epsilon \leqslant \tfrac{4}{9}(\mathbb{F}^2-1)^{\frac{3}{2}}. \tag{1.6}$$

The corresponding bounds for stationary, transcritical flow are

$$\mathbb{F}_1^2 = 1-(\tfrac{9}{2}\epsilon)^{\frac{2}{3}}, \quad \mathbb{F}_{\mathrm{u}}^2 = 1+(\tfrac{9}{4}\epsilon)^{\frac{2}{3}} \quad (\epsilon \ll 1). \tag{1.7a, b}$$

The corresponding wave drag is

$$\frac{D}{\rho U^2 H} = \frac{3}{2}\left(\frac{P}{\rho U^2 H}\right)^2 = \frac{3}{2}\left(\frac{\epsilon}{\mathbb{F}^2}\right)^2 \tag{1.8}$$

for $\mathbb{F} < \mathbb{F}_1$ and vanishes for $\mathbb{F} > \mathbb{F}_{\mathrm{u}}$.

The downstream cnoidal wave that appears for $\mathbb{F} < 1$ reduces to a sine wave in the limit $(1-\mathbb{F}^2)^{-\frac{3}{2}}\epsilon \to 0$ and to a transition to a uniform supercritical flow through a truncated solitary wave for $(1-\mathbb{F}^2)^{-\frac{3}{2}}\epsilon = \frac{2}{9}$, i.e. for $\mathbb{F} = \mathbb{F}_1$. The mean elevation is decreased for all cnoidal wavetrains in the lower transcritical regime $(1-\mathbb{F}_1 < 1-\mathbb{F} \ll 1)$, and the downstream Froude number tends to $1/\mathbb{F}_1$ as $\mathbb{F}$ tends to $\mathbb{F}_1$. This last result is confirmed in Appendix C through a straightforward balance of mass, impulse-momentum (which is decreased by wave drag) and energy.

2. Extension of Rayleigh's formulation

Following Rayleigh's (1876) formulation of the solitary-wave problem (Lamb 1932, §252),‡ we consider a steady flow in a reference frame moving with the force P, in which the upstream ($x \uparrow \infty$) depth and velocity are H and $(-U, 0)$, and derive the local velocity $\boldsymbol{q}$ from a stream function ψ according to

$$\boldsymbol{q} = (-\psi_y, \psi_x), \tag{2.1}$$

where (x, y) are Cartesian coordinates with origin at the bottom of the canal directly below P. The assumption of an inviscid, irrotational flow between the bottom ($y = 0$) and the free surface ($y = h$) then implies the kinematical boundary-value problem

$$\psi_{xx} + \psi_{yy} = 0 \quad (0 < y < h), \tag{2.2a}$$

$$\psi = 0 \quad (y = 0), \quad \psi = UH \equiv Q \quad (y = h). \tag{2.2b, c}$$

The corresponding Bernoulli equation at the free surface is

$$\tfrac{1}{2}q^2 + gh + \frac{p}{\rho} = \tfrac{1}{2}U^2 + gH \quad (y = h), \tag{2.3}$$

where $q \equiv |\boldsymbol{q}|$, p is the superficial pressure (which is assumed to vanish at $x = \infty$), and ρ is the fluid density.

† Wu & Wu (1982) and Ertekin (1984) obtain stationary, downstream wavetrains for $\mathbb{F} < 1$ through numerical integration of the Boussinesq and Green–Naghdi equations, respectively. Ertekin (see his figure 16) obtains the critical value $\mathbb{F} = 0.5$ for $\epsilon = 0.25$, which corresponds to (1.5) with $\frac{2}{9}$ replaced by 0.26. The difference between these last two numbers presumably stems from the retention of full nonlinearity in the Green–Naghdi model (see Appendix B).

‡ The development in this section is similar to that of Benjamin & Lighthill (1954) but provides for the superficial pressure p.

The solution of (2.2a, b) may be posed in the form

$$\psi = q_0(x)\,y - \tfrac{1}{6}q_0''(x)\,y^3 + O(\beta^2 Q), \tag{2.4}$$

where $q_0(x)$ ($= F'$ in Lamb's notation) is the particle speed at $y = 0$, $(\ )' \equiv \mathrm{d}(\ )/\mathrm{d}x$, and $\beta \equiv (H/L)^2$. It then follows from (2.2c) that

$$q_0 = Q\left\{1 + \tfrac{1}{6}h^2\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right] + O(\beta^2)\right\}h^{-1}. \tag{2.5}$$

Combining (2.1), (2.4) and (2.5) to express q^2 in (2.3) in terms of h and invoking $h'' = h'(\mathrm{d}h'/\mathrm{d}h)$, we obtain†

$$\tfrac{1}{2}Q^2\left[\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{h'^2}{h}\right) + \frac{1}{h^2}\right] + gh + \frac{p}{\rho} = \tfrac{1}{2}U^2 + gH. \tag{2.6}$$

Integrating (2.6) with respect to h and invoking $h = H$ at $x = \infty$, we obtain the energy equation

$$\frac{Q^2h'^2}{6h} = \frac{(U^2 - gh)(h-H)^2}{2h} + \frac{1}{\rho}\int_x^\infty ph'\,\mathrm{d}x, \tag{2.7}$$

which reduces to Lamb's (1932) §252 (8) for $p = 0$. The derivation of (2.6) and (2.7) from Hamilton's principle is given in Appendix A.

The vertically averaged, horizontal velocity deduced from (2.1) and (2.2) in the reference frame of the moving force is (positive to the left)

$$u = \frac{1}{h}\int_0^h \psi_y\,\mathrm{d}y = \frac{Q}{h}. \tag{2.8}$$

The corresponding, local Froude number is

$$\mathbb{F}_{\mathrm{h}} \equiv \frac{u}{(gh)^{\frac{1}{2}}} = \frac{Q}{(gh)^{\frac{3}{2}}} = \left(\frac{H}{h}\right)^{\frac{3}{2}}\mathbb{F}. \tag{2.9}$$

3. Compact forcing

We now suppose that the horizontal scale of p is small compared with the characteristic length L. The integral in (2.7) then may be approximated according to

$$\int_x^\infty ph'\mathrm{d}x \approx \boldsymbol{H}(-x)\int_{-\infty}^\infty ph'\,\mathrm{d}x = D\boldsymbol{H}(-x), \tag{3.1}$$

where $\boldsymbol{H}$ is Heaviside's step function and D is the wave drag, and (2.7) transforms to

$$\tfrac{1}{3}Q^2h'^2 = (U^2 - gh)(h-H)^2 + 2\left(\frac{D}{\rho}\right)h\boldsymbol{H}(-x) \equiv C_\pm(h) \quad (x \gtrless 0), \tag{3.2}$$

where $C_+(h)$ and $C_-(h)$ are cubic polynomials. It may be shown that (3.1), (3.2) and the subsequent results also hold for an equivalent bump of height $p/\rho g$ and horizontal lengthscale small compared with L.

It follows from (3.1) that, to the same approximation,

$$D = \tfrac{1}{2}P(h'_{0-} + h'_{0+}), \tag{3.3}$$

† Equation (2.6) is equivalent to Ertekin's (1984) 'Green–Naghdi equation' (5.25) after invoking $\overline{U} + \overline{u} = \mathbb{F}H/h$ and $\overline{U} = \mathbb{F}$ therein. See also Naghdi & Vongsarnpigoon's (1985) (2.3), which allows for variable depth.

where $P \equiv \rho g A$ is the net vertical force exerted by p, A is the cross-sectional area of the equivalent bump, and the subscript $0\pm$ signifies evaluation at $x = 0\pm$. Differencing (3.2) across $x = 0$ and comparing the result with (3.3), we obtain the identities

$$D = \tfrac{1}{6}\rho Q^2 h_0^{-1}(h_{0-}'^2 - h_{0+}'^2), \quad P = \tfrac{1}{3}\rho Q^2 h_0^{-1}(h_{0-}' - h_{0+}'). \qquad (3.4\,a, b)$$

4. Cnoidal waves ($\mathbb{F} < 1$)

The admissible solutions of (3.2), subject to (1.4), depend on the disposition of the zeros of $C_{\pm}(h)$ in $x \gtrless 0$ (cf. Benjamin & Lighthill 1954). If $\mathbb{F} < 1$ the simple zero of C_+ at $h = U^2/g = \mathbb{F}^2 H$ lies to the left of the double zero at $h = H$, in consequence of which the only non-trivial, bounded solution of (3.2) and (1.4) in $x > 0$ is $h = H$. It then follows that $h_0 = H$ and $h_{0+}' = 0$, and the elimination of h_{0-}' between (3.4a, b) on the hypothesis that a non-trivial solution exists in $x < 0$ yields

$$\frac{D}{\rho U^2 H} = \frac{3}{2}\left(\frac{P}{\rho U^2 H}\right)^2. \qquad (4.1)$$

This last result also may be deduced from the linear theory by letting $\alpha = \kappa h \to 0$ in Lamb's (1932) §245 (9), (19), (24) and §249 (3).

Cnoidal solutions of (3.2) exist in $x < 0$ if and only if $C_-(h)$ has three real zeros, which limits the parametric domain of ϵ and $\mathbb{F}$ (see Appendix B). Moreover, the antecedent requirement that dispersion be weak ($\beta \ll 1$ in §2) is satisfied for these cnoidal waves if and only if $0 < 1-\mathbb{F}^2 \ll 1$, in which domain both β and $(h-H)/H$ are $O(1-\mathbb{F}^2)$, $\epsilon = O[(1-\mathbb{F}^2)^{\frac{3}{2}}]$, and the condition that the three zeros be real reduces to (1.5) or, equivalently, $|\varpi| < \frac{2}{9}$, where

$$\alpha \equiv 1-\mathbb{F}^2 \ll 1, \quad \varpi \equiv \frac{A}{\alpha^{\frac{3}{2}} H^2} = O(1). \qquad (4.2\,a, b)$$

The construction of the cnoidal solutions for $|\varpi| < \frac{2}{9}$ is straightforward; however, I consider further only the limiting cases $|\varpi| \downarrow 0$ and $|\varpi| \uparrow \frac{2}{9}$.

Introducing the dimensionless, $O(1)$ variables

$$\xi = \frac{(3\alpha)^{\frac{1}{2}} x}{2\mathbb{F} H} \equiv \frac{x}{L}, \quad \eta(\xi) = \frac{h-H}{\alpha H}, \qquad (4.3\,a, b)$$

invoking (4.2a), solving the resulting reduction of (3.2) subject to

$$h = H \quad \text{and} \quad \operatorname{sgn} h' = \operatorname{sgn} P \quad \text{at } x = 0-,$$

and expanding the result about $\varpi = 0$, we obtain

$$\eta = \sqrt{3}\,\varpi \sin 2\xi + \tfrac{3}{4}\varpi^2(-3 + 4\cos 2\xi - \cos 4\xi) + O(\varpi^3) \quad (\xi < 0). \qquad (4.4)$$

The leading term in (4.4), $\sqrt{3}\,\varpi \sin 2\xi$, corresponds to linear theory with weak dispersion and may be obtained from Lamb's (1932) §245 (6) by approximating $kh \coth kh$ by $1 + \frac{1}{3}(kh)^2$ therein (Lamb's $h = H$ in the present notation). The mean value, $\langle\eta\rangle = -\frac{9}{4}\varpi^2$, agrees with that predicted by Benjamin (1970), wherein (Benjamin$\to$Miles) $a/h_0 = 3\alpha\varpi$, $\kappa h_0 = (3\alpha)^{\frac{1}{2}}$, $\gamma = 1-\alpha$, $\delta_+ = \frac{3}{2}\alpha\varpi^2$, $\delta_w = -\frac{3}{4}\alpha\varpi^2$, $\delta_- = -\frac{3}{32}\alpha^3\varpi^2$, $\delta_w - \delta_+ = \alpha\langle\eta\rangle$, all within $1 + O(\alpha)$, and

$$h_0 = H(1-\delta_+) = H(1-\tfrac{3}{2}\alpha\varpi^2) \qquad (4.5)$$

is the initial ($t = 0$) depth. Benjamin's upstream velocity (U in his notation) and upstream Froude number are $U(1+\frac{3}{2}\alpha\varpi^2)$ and $\mathbb{F}(1+\frac{9}{4}\alpha\varpi^2)$ in the present notation.

Note that Benjamin's calculation, being based on a sinusoidal wavetrain, is valid only for $|\varpi| \ll \frac{2}{9}$.

The limiting solution for $|\varpi| = \frac{2}{9}$ is given by

$$\eta = -\tfrac{2}{3}+\operatorname{sech}^2(\xi_0-\xi) \quad (\xi < 0), \quad \xi_0 = \tfrac{1}{2}(\operatorname{sgn}\varpi)\ln(2+\sqrt{3}). \tag{4.6a, b}$$

The limiting ($\xi\downarrow-\infty$) depth, velocity (positive to the left) in the reference frame of the force, and Froude number are

$$h_- = H(1-\tfrac{2}{3}\alpha), \quad u_- = U(1+\tfrac{2}{3}\alpha), \quad F_- = F(1+\alpha) = \frac{1}{F} \tag{4.7a, b, c}$$

to first order in α, and (4.6) describes a transition from a uniform, subcritical ($F < 1$) flow to a uniform, supercritical ($1/F > 1$) flow through a truncated solitary wave. An alternative derivation of (4.7) is given in Appendix C.

5. Cusped solitary waves ($F > 1$)

If $F > 1$ C_- has only one zero for $D > 0$, in consequence of which (3.2) and (1.4) admit a non-trivial solution only if $D = 0$ ($D < 0$ is physically inadmissible). It then is expedient to introduce ξ and η from (4.3), but with

$$\alpha = F^2-1 \tag{5.1}$$

in place of (4.2a), to reduce (3.2) to

$$\tfrac{1}{4}\eta'^2 = \eta^2-\eta^3. \tag{5.2}$$

The corresponding boundary conditions, obtained by transforming (1.4) and (3.4b), are

$$\eta\to 0 \quad (\xi\uparrow\infty), \quad \eta'_{0-}-\eta'_{0+} = 2\sqrt{3}\,\varpi. \tag{5.3a, b}$$

[$(1+\alpha\eta_0)/(1+\alpha)$ has been approximated by 1 in reducing (3.4b) to (5.3b), which is consistent with the approximation of weak dispersion, as in §4; see Appendix B.]

The only non-trivial solution of (5.2) and (5.3) is given by the symmetrical, cusped solitary wave (see figure 1)

$$\eta = \operatorname{sech}^2(|\xi|+\xi_0), \tag{5.4}$$

where ξ_0 is determined by

$$\operatorname{sech}^2\xi_0 \equiv \eta_0, \quad \eta_0^2(1-\eta_0) = \tfrac{3}{4}\varpi^2, \quad \operatorname{sgn}\xi_0 = \operatorname{sgn}\varpi. \tag{5.5a, b, c}$$

The cusp is convex/concave for $\varpi \gtrless 0$. The convexly cusped solitary wave for $\varpi > 0$ (figure 1a) resembles the asymptotic ($t\uparrow\infty$) results obtained by Wu & Wu (1982) through numerical integration of the Boussinesq equations with $F = 1.2$ and 1.4. The corresponding comparison for $\varpi < 0$ is less satisfactory in that the numerical results do not exhibit a concave centre (figure 1b).

It follows from (5.5b) that: the solution (5.4) is admissible if and only if $|\varpi| \leqslant \frac{4}{9}$, which implies the restriction (1.6); two such solutions are possible, with peak values that lie in $(0, \frac{2}{3})$ and $(\frac{2}{3}, 1)$ respectively, for each value of $|\varpi|$ in $(0, \frac{4}{9})$. The local Froude number, as calculated from (2.9), is

$$F_\eta = 1+\tfrac{1}{2}\alpha(1-3\eta)+O(\alpha^2) \tag{5.6}$$

and has a minimum (at $\xi = 0$) for the convexly cusped wave that decreases from $1+\frac{1}{2}\alpha$ at $\eta_0 = 0$ ($\varpi = 0$) through $1-\frac{1}{2}\alpha = 1/F$ at $\eta_0 = \frac{2}{3}$ ($\varpi = \frac{4}{9}$) to $1-\alpha$ at $\eta_0 = 1$ ($\varpi = 0$).

The limit $\eta_0\downarrow\frac{1}{2}\sqrt{3}\,\varpi\downarrow 0$ ($\xi_0\uparrow\infty$), which corresponds to linear, shallow-water theory (Lamb (1932, §177), yields

$$\eta = \tfrac{1}{2}\sqrt{3}\,\varpi\,\mathrm{e}^{-2|\xi|} \quad (\varpi\downarrow 0). \tag{5.7}$$

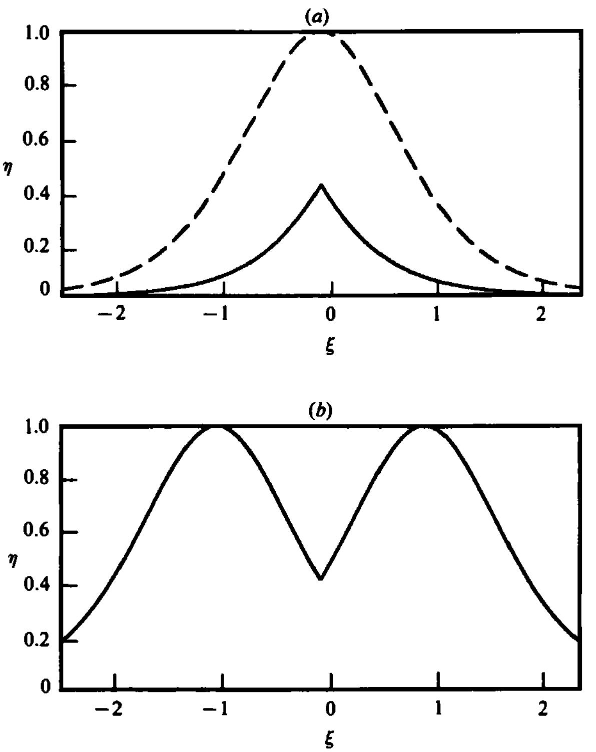

FIGURE 1. The cusped solitary wave: (*a*) $\xi_0 = 0$ (– – –), $\xi_0 = 1$ (——); (*b*) $\xi_0 = -1$.

The limit $\varpi \uparrow 0$, for which the maximum displacement remains $\eta = 1$ at $|\xi| = |\xi_0|$, is inaccessible through linear theory, which implies $\eta < 0$ for $\varpi < 0$; this suggests that the concavely cusped solitary wave may be either unstable or unrealizable in a real fluid.

The limit $1-\eta_0 \downarrow \frac{3}{4}\varpi^2 \downarrow 0$ ($|\xi_0| \to 0$) yields the free solitary wave

$$\eta = \mathrm{sech}^2\,\xi \quad (\varpi = 0). \tag{5.8}$$

A convincing determination of which of the two solitary waves admitted by (5.5) is realized for $|\varpi| \leqslant \frac{4}{9}$ presumably requires the solution of the corresponding initial-value problem (or a stability analysis) and is beyond the purview of the steady-state model (in particular, the limits $t \uparrow \infty$ and $\varpi \downarrow 0$ may not be commutative). The analogy with weak and strong shock waves in a gas suggests that the weaker solution would be realized for $\varpi > 0$, at least for motion starting from rest, but the fact that the free solitary wave (5.8) can exist renders this argument less than compelling.

The dimensionless volumetric displacement implied by (5.4) is

$$\int_{-\infty}^{\infty} \eta \, \mathrm{d}\xi = 2(1-\tanh \xi_0), \tag{5.9}$$

which presumably is balanced by a disturbance of net negative volumetric displacement that moves to $\xi = -\infty$ (downstream of the moving force) during the evolution of (5.4) from a configuration of rest ($\eta = 0$). This balancing disturbance could be of infinitesimal amplitude and infinite extent.

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Appendix A. Lagrangian formulation

The Lagrangian for the steady flow postulated in §2 is given by

$$L = T - V + W \equiv \int_{-\infty}^{\infty} \mathscr{L}\rho \, \mathrm{d}x, \tag{A 1}$$

where

$$T = \tfrac{1}{2}\rho \int_{-\infty}^{\infty} \mathrm{d}x \int_0^h [\psi_x^2 + (\psi_y - U)^2] \, \mathrm{d}y \tag{A 2}$$

is the kinetic energy in the laboratory reference frame,

$$V = \tfrac{1}{2}\rho g \int_{-\infty}^{\infty} (h-H)^2 \, \mathrm{d}x \tag{A 3}$$

is the potential energy,

$$W = -\int_{-\infty}^{\infty} p(h-H) \, \mathrm{d}x \tag{A 4}$$

is the work done by the superficial pressure, and $\mathscr{L}$ is the Lagrangian density.

Substituting ψ from (2.4) into (A 2), invoking (2.5), carrying out the integration with respect to y, and neglecting $O(\beta^2)$, we obtain

$$T = \tfrac{1}{2}\rho Q^2 \int_{-\infty}^{\infty} \left[\tfrac{1}{3}h'^2 + \left(\frac{h-H}{H}\right)^2\right] \frac{\mathrm{d}x}{h}. \tag{A 5}$$

Combining (A 3)–(A 5) in (A 1), we obtain

$$\mathscr{L} = \frac{Q^2 h'^2}{6h} + \frac{(U^2 - gh)(h-H)^2}{2h} - \frac{p}{\rho}(h-H). \tag{A 6}$$

The requirement that $\mathscr{L}$ be stationary with respect to variations of h (Hamilton's principle),

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathscr{L}}{\partial h'} - \frac{\partial \mathscr{L}}{\partial h} = 0, \tag{A 7}$$

then yields (2.6).

It is evident from (A 1) and (A 7) that (A 6) may be regarded as the Lagrangian for a single-degree-of-freedom system in which x and $\mathscr{L}/\rho$ are analogues of time and action. The corresponding Hamiltonian is

$$\mathscr{H} = h'\frac{\partial \mathscr{L}}{\partial h'} - \mathscr{L} = \frac{Q^2 h'^2}{6h} - \frac{(U^2 - gh)(h-H)^2}{2h} + \frac{p}{\rho}(h-H). \tag{A 8}$$

The total derivative of $\mathscr{H}$, *qua* function of h, h' and x, then is given by (cf. Landau & Lifshitz 1969)

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{\partial H}{\partial x} = \frac{p'}{\rho}(h-H). \tag{A 9}$$

Integrating (A 9) with respect to x and invoking (1.4), which implies $\mathscr{H} = 0$ at $x = \infty$, we obtain

$$\mathscr{H} = \frac{1}{\rho}\int_{\infty}^{x} p'(h-H)\,\mathrm{d}x = \frac{p}{\rho}(h-H) + \frac{1}{\rho}\int_x^{\infty} ph'\,\mathrm{d}x, \tag{A 10}$$

which, in conjunction with (A 8), implies (2.7).

Appendix B. Green–Naghdi model

The Green–Naghdi model (Naghdi & Vongsarnpigoon 1985), which, in my view, implicitly assumes weak dispersion as in §2† but does not assume weak nonlinearity ($|h-H| \ll H$), yields (2.6), which is equivalent to Ertekin's (1984) equation (5.25). The assumptions of the upstream null condition (1.4) and a compact pressure distribution then lead to (3.2), which may be integrated without further approximations to obtain counterparts of the results in §§4 and 5.

Considering first $F < 1$, we introduce

$$x = \frac{\beta^{\frac{1}{2}} x}{H} \equiv \frac{x}{L}, \quad h = \frac{h}{H} \tag{B 1a, b}$$

and reduce (3.2) to

$$\tfrac{1}{3}\beta F^2 \left(\frac{dh}{dx}\right)^2 = (F^2 - h)(h-1)^2 + 3\left(\frac{\epsilon}{F}\right)^2 h \quad (x < 0) \tag{B 2a}$$

$$\equiv (a-h)(h-b)(h-c). \tag{B 2b}$$

The differential equation (B 2) may be integrated, subject to $h = 1$ at $x = 0$, to obtain a real, bounded solution if and only if the right-hand side, *qua* cubic in h, has three real zeros, which, in turn, requires

$$\epsilon < \epsilon_* = F\left[\frac{8+20F^2 - F^4 - F(8+F^2)^{\frac{3}{2}}}{24}\right]^{\frac{1}{2}} \equiv \tfrac{2}{9}(1-F^2)^{\frac{3}{2}} \Phi(F), \tag{B 3}$$

where Φ increases from $\Phi(0) = 0$ to $\Phi(1) = 1$ (see figure 2). The solution then is

$$h = b + (a-b)\,\mathrm{cn}^2(x_0 - x; k), \quad k^2 = \frac{a-b}{a-c}, \tag{B 4a, b}$$

with

$$\beta = \frac{3}{4}\left(\frac{a-c}{F^2}\right), \tag{B 5}$$

where cn ( ) is an elliptic cosine of modulus k, and the constant of integration x_0 is determined by $h = 1$ at $x = 0$. Note that the characteristic length L now is defined by (B 1a) and (B 5), rather than (4.3a), and that a and c, and hence β, are determined as functions of F and ϵ by the identity between the right-hand sides of (B 2a) and (B 2b).

The limiting values of β are given by

$$\beta = \frac{3}{4}\left(\frac{1-F^2}{F^2}\right) + O\left[\frac{\epsilon^2}{(1-F^2)^2}\right] \quad (\epsilon^2 \downarrow 0) \tag{B 6a}$$

and

$$\beta_* = \frac{3}{4F^2}\left[(1-F^2)^2 + \left(\frac{3\epsilon_*}{F}\right)^2\right]^{\frac{1}{2}} \quad (\epsilon = \epsilon_*). \tag{B 6b}$$

This suggests, and more detailed calculations confirm, that β is small (as has been assumed in §2) if and only if $1 - F^2 \ll 1$, in which case

$$\epsilon_* = \tfrac{2}{9}(1-F^2)^{\frac{3}{2}}, \quad \beta = \tfrac{3}{4}(1-F^2), \tag{B 7a, b}$$

within $1 + O(1-F^2)$, as in §4.

The solution of (3.2) and (1.4) for $F > 1$ is given by (5.4) without further approximation, but the exact invocation of (3.4b) yields

$$\epsilon^2 = \frac{4}{3}\frac{F^2(F^2-1)^3 \eta_0^2 (1-\eta_0)}{[1+(F^2-1)\eta_0]^2} \tag{B 8}$$

† Professor Naghdi (private communication) does not agree with this assertion. See also Miles & Salmon (1985).

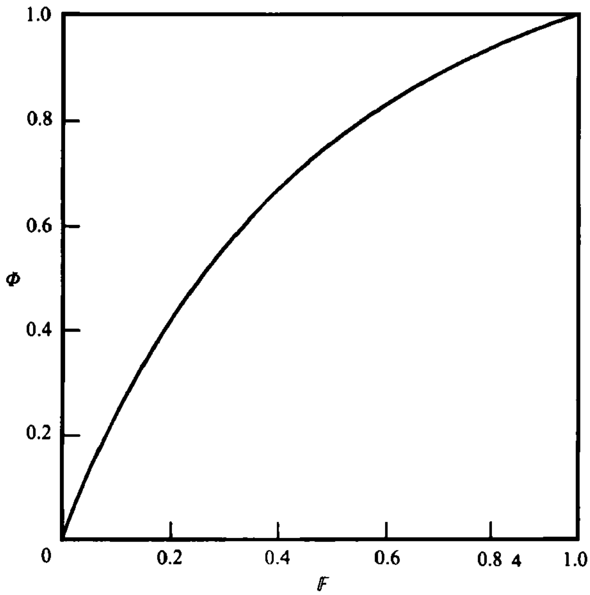

FIGURE 2. The function $\Phi(\mathbb{F})$, as defined by (B 3).

in place of (5.5*b*). Maximizing the right-hand side of (B 8), we obtain

$$\epsilon_* = \tfrac{4}{9}(\mathbb{F}^2-1)^{\frac{3}{2}}\Phi(1/\mathbb{F}), \tag{B 9}$$

where Φ is defined by (B 3). The counterpart of (B 5) is (cf. (B 6))

$$\beta = \frac{3}{4}\left(\frac{\mathbb{F}^2-1}{\mathbb{F}^2}\right), \tag{B 10}$$

which is small if and only if $\mathbb{F}^2-1 \ll 1$.

Appendix C. Transition between uniform flows

The result (4.7) also may be derived through a slight variation of Rayleigh's (1914) analysis for a transition between two uniform levels. (Impulse + momentum, but not energy, is conserved in Rayleigh's calculation; the converse is true in the present calculation.) Following Lamb (1932, §187), but reversing the direction of flow, we let $h_+ \equiv H$ and $u_+ \equiv U$ be the upstream depth and velocity (positive to the left) in the reference frame of the force and let h_- and u_- be the corresponding downstream quantities. The equations of mass, momentum and energy then are

$$h_- u_- = h_+ u_+ \equiv Q, \tag{C 1}$$

$$\rho Q(u_- - u_+) = \tfrac{1}{2}\rho g(h_+^2 - h_-^2) - D, \tag{C 2}$$

$$\rho Q(\tfrac{1}{2}u_-^2 + gh_-) = \rho Q(\tfrac{1}{2}u_+^2 + gh_+). \tag{C 3}$$

Substituting u_- and u_+ from (C 1) into (C 2) and (C 3) and eliminating Q between the resulting equations, we obtain

$$(h_+ - h_-)^3 = \frac{2D(h_- + h_+)}{\rho g} = \frac{4DH}{\rho g}[1 + O(\alpha)]. \tag{C 4}$$

Substituting $D = \frac{3}{2}(\epsilon/\mathbb{F})^2 \rho g H^2$ from (1.8) and invoking $\epsilon^2 = (\frac{4}{9})^2 \alpha^3$ we obtain

$$(h_+ - h_-)^3 = (\tfrac{2}{3}\alpha H)^3 [1 + O(\alpha)], \tag{C 5}$$

which is equivalent to (4.7*a*); (4.7*b*) then follows from (C 1).

If energy dissipation is introduced by inserting the term $-W$ on the right-hand side of (C 3), (C 4) becomes

$$(h_+ - h_-)^3 = \frac{4H}{\rho g}\left[D - \frac{W}{(gH)^{\frac{1}{2}}}\right][1 + O(\alpha)], \tag{C 6}$$

which is equivalent to Rayleigh's result if $D = 0$.

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